

Hadronic vacuum polarization function within dispersive approach to QCD

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The dispersive approach to quantum chromodynamics is applied to the study of the hadronic vacuum polarization function and associated quantities. This approach merges the intrinsically nonperturbative constraints, which originate in the kinematic restrictions on the respective physical processes, with corresponding perturbative input. The obtained hadronic vacuum polarization function agrees with pertinent lattice simulation data. The evaluated hadronic contributions to the muon anomalous magnetic moment and to the shift of the electromagnetic fine structure constant conform with recent assessments of these quantities.

PACS numbers: 11.55.Fv, 12.38.Lg, 13.40.Em, 14.60.Ef

I. INTRODUCTION

The theoretical description of a number of the strong interaction processes is inherently based on the hadronic vacuum polarization function $\Pi(q^2)$. In particular, this function plays a crucial role in the studies of inclusive τ lepton hadronic decay and of electron-positron annihilation into hadrons, that provides decisive self-consistency tests of quantum chromodynamics (QCD). At the same time, the function $\Pi(q^2)$ enters in the analysis of the hadronic contributions to such quantities of precise particle physics as the muon anomalous magnetic moment and the running of the electromagnetic fine structure constant, that, in turn, puts strong limits on the effects due to a possible new fundamental physics beyond the standard model (SM). Additionally, the theoretical exploration of the aforementioned processes constitutes a natural framework for a thorough investigation of both perturbative and intrinsically nonperturbative aspects of hadron dynamics.

The strong interactions possess the feature of the asymptotic freedom, that makes it possible to apply perturbation theory to the study of ultraviolet behaviour of the function $\Pi(q^2)$. However, there is still no rigorous method of theoretical description of hadron dynamics at low energies, which would have provided one with robust unabridged results. This fact eventually forces one to engage a variety of nonperturbative approaches in order to examine the strong interactions in the infrared domain. For example, an insight into the low-energy behaviour of the hadronic vacuum polarization function can be gained from such methods as, e.g., lattice simulations [1–4], operator product expansion [5–10], instanton liquid model [11, 12], and others.

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Theoretical particle physics widely employs various methods based on the dispersion relations¹. In particular, the latter provide a source of the nonperturbative information about the low-energy hadron dynamics. Specifically, the dispersion relations, which render the kinematic restrictions on the relevant physical processes into the mathematical form, impose stringent constraints on the pertinent quantities [such as $\Pi(q^2)$ and related functions], that should certainly be accounted for when one comes out of the applicability range of perturbation theory. These nonperturbative constraints have been merged with corresponding perturbative input in the framework of dispersive approach to QCD [30, 31], which provides unified integral representations for the functions on hand, see Sec. II.

The primary objective of this paper is to calculate the hadronic vacuum polarization function within dispersive approach and to compare it with relevant lattice simulation data, as well as to evaluate the corresponding hadronic contributions to the muon anomalous magnetic moment and to the shift of the electromagnetic fine structure constant.

The layout of the paper is as follows. In Sec. II the dispersive approach to QCD [30, 31] is overviewed. Section III presents the comparison of the hadronic vacuum polarization function calculated in the framework of dispersive approach with pertinent lattice simulation data and elucidates the qualitative distinctions between the approach on hand, its massless limit, and perturbative approach. Section IV contains the evaluation of hadronic contributions to the aforementioned electroweak observables. In the Conclusions (Sect. V) the basic results are summarized and further studies within this approach are outlined. Auxiliary material is given in the Appendix.

II. DISPERSIVE APPROACH TO QUANTUM CHROMODYNAMICS

The hadronic vacuum polarization function $\Pi(q^2)$ is defined as the scalar part of the hadronic vacuum polarization tensor

$$\Pi_{\mu\nu}(q^2) = i \int d^4x e^{iqx} \langle 0 | T \{ J_\mu(x) J_\nu(0) \} | 0 \rangle = \frac{i}{12\pi^2} (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(q^2). \quad (1)$$

The kinematics of the process on hand determines the cut structure of $\Pi(q^2)$ in the complex q^2 -plane. Specifically, the function $\Pi(q^2)$ (1) has the only cut along the positive semiaxis of real q^2 starting at the hadronic production threshold $4m_\pi^2 = m^2$ (discussion of this issue can be found in, e.g., Ref. [32], as well as in Refs. [30, 31, 33]). Proceeding from this fact and bearing in mind the asymptotic ultraviolet behaviour of the hadronic vacuum polarization function one can write down the corresponding dispersion relation by making use of the once-subtracted Cauchy integral formula [see Eq. (2) below]. For practical purposes it proves to be convenient to define the Adler function $D(Q^2)$ [34] [see Eq. (6) below] and the related function $R(s)$, which is identified with the so-called R -ratio of electron-positron annihilation into hadrons [see Eq. (4) below]. Eventually, the complete set of well-known relations, which express the functions $\Pi(q^2)$, $R(s)$, and $D(Q^2)$ in terms of each other, acquires the following

¹ Among the recent applications of such methods are, for example, the extension of applicability range of chiral perturbation theory [13, 14], the precise determination of parameters of resonances [15], the assessment of the hadronic light-by-light scattering [16], and many others [17–29].

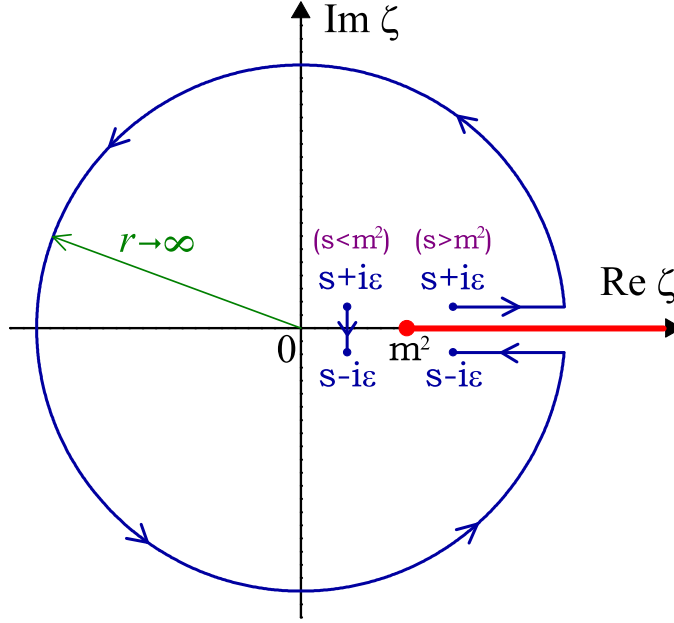


FIG. 1. The integration contour in Eq. (5). The physical cut $\zeta \geq m^2$ of the Adler function $D(-\zeta)$ (6) is shown along the positive semiaxis of real ζ .

form (see papers [34–36] as well as [31] and references therein):

$$\Delta\Pi(q^2, q_0^2) = (q^2 - q_0^2) \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma - q^2)(\sigma - q_0^2)} d\sigma \quad (2)$$

$$= - \int_{-q_0^2}^{-q^2} D(\zeta) \frac{d\zeta}{\zeta}, \quad (3)$$

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} \Delta\Pi(s + i\varepsilon, s - i\varepsilon) \quad (4)$$

$$= \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta}, \quad (5)$$

$$D(Q^2) = - \frac{d\Pi(-Q^2)}{d \ln Q^2} \quad (6)$$

$$= Q^2 \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma + Q^2)^2} d\sigma. \quad (7)$$

In Eqs. (2)–(7) $\Delta\Pi(q^2, q_0^2) = \Pi(q^2) - \Pi(q_0^2)$, whereas $Q^2 = -q^2 > 0$ and $s = q^2 > 0$ denote the spacelike and timelike kinematic variables, respectively. The common prefactor $N_c \sum_{f=1}^{n_f} Q_f^2$ is omitted throughout the paper, where $N_c = 3$ is the number of colours, Q_f stands for the electric charge of f -th quark (in units of the elementary charge e), and n_f denotes the number of active flavours. The integration contour in Eq. (5) lies in the region of analyticity of its integrand (see Fig. 1). Note that the derivation of relations (2)–(7) requires the knowledge of the cut structure of hadronic vacuum polarization function $\Pi(q^2)$ (1)

and its asymptotic ultraviolet behaviour. It is worth mentioning also that Eqs. (2) and (7) can be used for extracting the functions $\Delta\Pi(q^2, q_0^2)$ and $D(Q^2)$ from the experimental data on $R(s)$.

As noted in the Introduction, the dispersion relations (2)–(7) embody the kinematic restrictions on the respective physical processes and impose intrinsically nonperturbative constraints on the functions $\Pi(q^2)$, $R(s)$, and $D(Q^2)$, that should certainly be taken into account when one oversteps the limits of applicability of perturbation theory. These non-perturbative constraints² have been merged with corresponding perturbative input in the framework of dispersive approach to QCD³ [30, 31], which provides the following unified integral representations for the functions on hand:

$$\Delta\Pi(q^2, q_0^2) = \Delta\Pi^{(0)}(q^2, q_0^2) + \int_{m^2}^{\infty} \rho(\sigma) \ln\left(\frac{\sigma - q^2}{\sigma - q_0^2} \frac{m^2 - q_0^2}{m^2 - q^2}\right) \frac{d\sigma}{\sigma}, \quad (8)$$

$$R(s) = R^{(0)}(s) + \theta(s - m^2) \int_s^{\infty} \rho(\sigma) \frac{d\sigma}{\sigma}, \quad (9)$$

$$D(Q^2) = D^{(0)}(Q^2) + \frac{Q^2}{Q^2 + m^2} \int_{m^2}^{\infty} \rho(\sigma) \frac{\sigma - m^2}{\sigma + Q^2} \frac{d\sigma}{\sigma}. \quad (10)$$

These equations have been obtained by employing the relations (2)–(7) and the asymptotic ultraviolet behaviour of the hadronic vacuum polarization function. In Eqs. (8)–(10) $\rho(\sigma)$ is the spectral density

$$\begin{aligned} \rho(\sigma) &= \frac{1}{2\pi i} \frac{d}{d \ln \sigma} \lim_{\varepsilon \rightarrow 0_+} \left[p(\sigma - i\varepsilon) - p(\sigma + i\varepsilon) \right] \\ &= -\frac{d}{d \ln \sigma} r(\sigma) \\ &= \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} \left[d(-\sigma - i\varepsilon) - d(-\sigma + i\varepsilon) \right], \end{aligned} \quad (11)$$

$p(q^2)$, $r(s)$, and $d(Q^2)$ denote the strong corrections to the functions $\Pi(q^2)$, $R(s)$, and $D(Q^2)$, respectively, $\theta(x)$ stands for the unit step-function [$\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ otherwise], and the leading-order terms read [32, 38]:

$$\Delta\Pi^{(0)}(q^2, q_0^2) = 2 \frac{\varphi - \tan \varphi}{\tan^3 \varphi} - 2 \frac{\varphi_0 - \tan \varphi_0}{\tan^3 \varphi_0}, \quad (12)$$

$$R^{(0)}(s) = \theta(s - m^2) \left(1 - \frac{m^2}{s} \right)^{3/2}, \quad (13)$$

$$D^{(0)}(Q^2) = 1 + \frac{3}{\xi} \left[1 - \sqrt{1 + \xi^{-1}} \sinh^{-1}(\xi^{1/2}) \right], \quad (14)$$

where $\sin^2 \varphi = q^2/m^2$, $\sin^2 \varphi_0 = q_0^2/m^2$, and $\xi = Q^2/m^2$, see papers [30, 31] and references therein for the details.

² Including the correct analytic properties in the kinematic variable, that implies the absence of unphysical singularities in Eqs. (8)–(10), see Sec. II A of Ref. [31] for the details.

³ Its preliminary formulation was discussed in Refs. [33, 37].

It is worthwhile to outline that the Adler function obtained in the framework of the dispersive approach⁴ (10) agrees with its experimental prediction in the entire energy range, see Refs. [30, 46, 47]. At the same time, the representations (8)–(10) conform with the results of Bethe–Salpeter calculations [48] as well as of lattice simulations [49]. Additionally, the dispersive approach has proved to be capable of describing OPAL (update 2012, Ref. [50]) and ALEPH (update 2014, Ref. [51]) experimental data on inclusive τ lepton hadronic decay in vector and axial–vector channels in a self–consistent way [31, 52] (see also Refs. [53, 54]).

In the framework of the approach on hand the corresponding perturbative input is accounted for in the same way as in other similar approaches, namely, by means of the spectral density (11). Specifically, the latter is approximated by its perturbative part, which can be calculated by making use of the perturbative expression for either of the strong corrections $p(q^2)$, $r(s)$, and $d(Q^2)$, see, e.g., papers [55, 56] and references therein:

$$\begin{aligned}\rho_{\text{pert}}(\sigma) &= \frac{1}{2\pi i} \frac{d}{d \ln \sigma} \lim_{\varepsilon \rightarrow 0_+} \left[p_{\text{pert}}(\sigma - i\varepsilon) - p_{\text{pert}}(\sigma + i\varepsilon) \right] \\ &= -\frac{d}{d \ln \sigma} r_{\text{pert}}(\sigma) \\ &= \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} \left[d_{\text{pert}}(-\sigma - i\varepsilon) - d_{\text{pert}}(-\sigma + i\varepsilon) \right].\end{aligned}\tag{15}$$

It is worth noting here that in the massless limit ($m^2 = 0$) for the case of perturbative spectral function (15) Eqs. (9) and (10) become identical to those of the analytic perturbation theory (APT) [17] (see also Refs. [18–29]). However, as discussed in Refs. [30, 31, 47, 54], the massless limit loses some of the substantial nonperturbative constraints, which relevant dispersion relations impose on the functions on hand, that appears to be essential for the studies of hadron dynamics at low energies.

III. COMPARISON OF $\Pi(q^2)$ WITH LATTICE SIMULATION DATA

As mentioned above, the lattice QCD simulations constitute an efficient method of investigation of the nonperturbative aspects of strong interactions. Over past time this method has been applied to an extensive study of a broad range of topics (for a recent overview see, e.g., Ref. [57]), including the low–energy behaviour of the hadronic vacuum polarization function $\Pi(q^2)$. It is of a particular interest to juxtapose the function $\Pi(q^2)$ obtained within dispersive approach (8) with relevant lattice simulation data.

To calculate the hadronic vacuum polarization function, it is convenient to proceed with the subtracted at zero form of Eq. (8), namely

$$\bar{\Pi}(Q^2) = \Delta\Pi(0, -Q^2) = \Delta\Pi^{(0)}(0, -Q^2) + \int_{m^2}^{\infty} \rho(\sigma) \ln\left(\frac{1 + Q^2/m^2}{1 + Q^2/\sigma}\right) \frac{d\sigma}{\sigma}.\tag{16}$$

In what follows we shall employ the perturbative expression for the spectral function (15). At the one–loop level it assumes a simple form [namely, $\rho_{\text{pert}}^{(1)}(\sigma) = (4/\beta_0)[\ln^2(\sigma/\Lambda^2) + \pi^2]^{-1}$,

⁴ The studies of the Adler function within other approaches can be found in, e.g., Refs. [39–45].

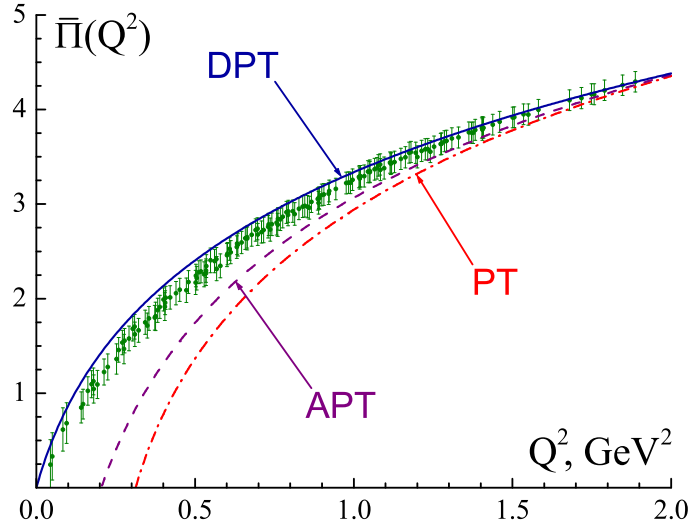


FIG. 2. Comparison of the four-loop hadronic vacuum polarization function calculated within dispersive approach (16) (solid curve) with lattice simulation data [59] (circles). The massless prediction of $\Pi(q^2)$ (17) is denoted by dashed curve, whereas its perturbative approximation (18) is shown by dot-dashed curve. Values of parameters: $\Lambda = 419 \text{ MeV}$, $n_f = 2$.

where $\beta_0 = 11 - 2n_f/3$ and Λ denotes the QCD scale parameter], whereas at the higher loop levels $\rho_{\text{pert}}(\sigma)$ (15) is rather cumbersome. The explicit expressions for the spectral function (15) up to the four-loop level⁵ can be found in Ref. [55].

As one can infer from Fig. 2, the hadronic vacuum polarization function (16) (solid curve), which was obtained within dispersively improved perturbation theory (DPT) delineated in the previous Section, is in a good agreement with lattice simulation data [59] (circles) (the rescaling procedure described in Refs. [60, 61] was applied). The presented result corresponds to the four-loop level and $n_f = 2$ active flavours. To elucidate the qualitative distinctions between the approaches mentioned in the previous Section, Fig. 2 also displays the one-loop Eq. (8) in the massless limit, which, in the considered case, corresponds to APT (dashed curve)

$$\Delta\Pi_{\text{APT}}^{(1)}(-Q^2, -Q_0^2) = \Delta\Pi_{\text{pert}}^{(0)}(-Q^2, -Q_0^2) - \frac{4}{\beta_0} \ln \left[\frac{a_{\text{an}}^{(1)}(Q_0^2)}{a_{\text{an}}^{(1)}(Q^2)} \right], \quad (17)$$

and the one-loop perturbative approximation of $\Pi(q^2)$ (dot-dashed curve)

$$\Delta\Pi_{\text{pert}}^{(1)}(-Q^2, -Q_0^2) = \Delta\Pi_{\text{pert}}^{(0)}(-Q^2, -Q_0^2) - \frac{4}{\beta_0} \ln \left[\frac{a_{\text{pert}}^{(1)}(Q_0^2)}{a_{\text{pert}}^{(1)}(Q^2)} \right]. \quad (18)$$

In these equations the leading-order terms read

$$\Delta\Pi_{\text{pert}}^{(0)}(-Q^2, -Q_0^2) = -\ln \left(\frac{Q^2}{Q_0^2} \right), \quad (19)$$

⁵ Recently completed calculation of the respective four-loop perturbative coefficient is given in Ref. [58].

the notation $a(Q^2) = \alpha(Q^2)\beta_0/(4\pi)$ is used,

$$\alpha_{\text{pert}}^{(1)}(Q^2) = \frac{4\pi}{\beta_0} \frac{1}{\ln z}, \quad z = \frac{Q^2}{\Lambda^2} \quad (20)$$

denotes the one-loop perturbative running coupling, and

$$\alpha_{\text{an}}^{(1)}(Q^2) = \frac{4\pi}{\beta_0} \frac{z-1}{z \ln z} \quad (21)$$

stands for the one-loop infrared enhanced analytic running coupling. It is interesting to note here that the expression (21) was first obtained in Refs. [62, 63] and has been independently rediscovered (proceeding from entirely different reasoning) later on in Ref. [64], see also Ref. [65].

As mentioned above, the dispersion relations (2)–(7) impose intrinsically nonperturbative constraints on the functions on hand, whereas the integral representations (8)–(10) merge these constraints with corresponding perturbative input. For example, as discussed in Refs. [30, 31], the relation (7) implies that the Adler function $D(Q^2)$ possesses the only cut $Q^2 \leq -m^2$ along the negative semiaxis of real Q^2 and that $D(Q^2)$ vanishes in the infrared limit $Q^2 \rightarrow 0$ (this condition holds for $m \neq 0$ only and appears to be lost in the massless limit). In turn, the first of these constraints indicates that the Adler function (10) contains no unphysical singularities, whereas the second one substantially stabilizes its infrared behaviour, see Refs. [30, 31]. Similarly, relation (2) signifies that the hadronic vacuum polarization function $\Pi(q^2)$ possesses the only cut $q^2 \geq m^2$ along the positive semiaxis of real q^2 and that the subtraction point q_0^2 can be located anywhere in the complex q^2 -plane except for this cut. In turn, the first of these constraints means that the hadronic vacuum polarization function (8) is free of the unphysical singularities, whereas the second one enables one to subtract $\Pi(q^2)$ at $q_0^2 = 0$ (for $m \neq 0$ only), that binds the low-energy behaviour of $\bar{\Pi}(Q^2)$ (16). This issue is illustrated by Fig. 2. Specifically, the perturbative approximation of the hadronic vacuum polarization function (18) (dot-dashed curve) contains infrared unphysical singularities, that makes it inapplicable at low energies. At the same time, although both expressions (16) and (17) are free of the unphysical singularities, their infrared behaviour is quite different. Namely, the hadronic vacuum polarization function (16) (solid curve) vanishes in the infrared limit, whereas the APT prediction (17) diverges at $Q^2 \rightarrow 0$. The latter originates in the mathematical fact that in the massless limit the function $\Pi(q^2)$ is undefined at the beginning of its branch cut. This makes the massless APT prediction of $\Pi(q^2)$ also incompatible with lattice simulation data at low energies. It is worthwhile to note also that the aforementioned features are universal and determine the qualitative behaviour of the hadronic vacuum polarization function within each of the approaches discussed above.

IV. HADRONIC CONTRIBUTIONS TO ELECTROWEAK OBSERVABLES

A. Muon anomalous magnetic moment

The theoretical description of the muon anomalous magnetic moment $a_\mu = (g_\mu - 2)/2$ is a long-standing challenging issue of the elementary particle physics, which engages the

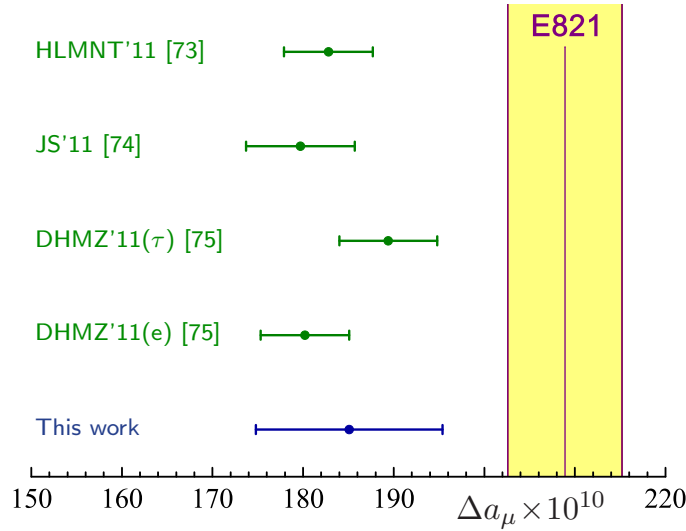


FIG. 3. Comparison of the subtracted muon anomalous magnetic moment (25) with its recent assessments [73–75] (circles). The averaged experimental value (26) is shown by vertical shaded band, $\Delta a_\mu = a_\mu - a_0$, and $a_0 = 11659 \times 10^{-7}$.

entire pattern of interactions within SM. Both experimental measurements [66, 67] and theoretical evaluations [68, 69] of a_μ have achieved an impressive accuracy, and the remaining discrepancy of the order of few standard deviations between them may be an evidence for the existence of a new physics beyond SM. The uncertainty of theoretical estimation of a_μ is mainly dominated by the leading-order hadronic contribution a_μ^{HLO} , which involves the integration of hadronic vacuum polarization function $\Pi(q^2)$ over the range inaccessible within perturbation theory⁶ (see, e.g., Ref. [70]):

$$a_\mu^{\text{HLO}} = \frac{1}{3} \left(\frac{\alpha}{\pi} \right)^2 \int_0^\infty f\left(\frac{\zeta}{4m_\mu^2}\right) \bar{\Pi}(\zeta) \frac{d\zeta}{4m_\mu^2} = \frac{1}{3} \left(\frac{\alpha}{\pi} \right)^2 \int_0^1 (1-x) \bar{\Pi}\left(m_\mu^2 \frac{x^2}{1-x}\right) dx. \quad (22)$$

In this equation

$$f(x) = \frac{1}{x^3} \frac{y^5(x)}{1-y(x)} \quad (23)$$

and $y(x) = x(\sqrt{1+x^{-1}} - 1)$ is a monotonously nondecreasing function of its argument, $0 \leq y(x) < 1/2$.

As mentioned above, in the framework of dispersive approach the hadronic vacuum polarization function $\Pi(q^2)$ (8) is free of the unphysical singularities, that enables one to perform the integration in Eq. (22) in a straightforward way [i.e., without involving experimental data on $R(s)$]. Thus, to evaluate a_μ^{HLO} within approach on hand, we shall employ Eqs. (22) and (16) with the four-loop spectral function (15), that eventually results in

$$a_\mu^{\text{HLO}} = (696.1 \pm 9.5) \times 10^{-10}. \quad (24)$$

⁶ To obviate this difficulty one can express a_μ^{HLO} (22) in terms of $R(s)$ by making use of Eq. (2) and replace the low-energy behaviour of $R(s)$ with relevant experimental data, see reviews [68, 69] and references therein.

In this equation the quoted error accounts for the uncertainties of the parameters entering Eq. (22), their values being taken from Refs. [71, 72]. The obtained estimation of the leading-order hadronic contribution to a_μ (24) appears to be in a good agreement with its recent assessments, namely, $a_\mu^{\text{HLO}} = (694.9 \pm 4.3) \times 10^{-10}$ (Ref. [73]), $a_\mu^{\text{HLO}} = (691.0 \pm 4.7) \times 10^{-10}$ (Ref. [74]), $a_\mu^{\text{HLO}} = (701.5 \pm 4.7) \times 10^{-10}$ (τ -based) and $a_\mu^{\text{HLO}} = (692.3 \pm 4.2) \times 10^{-10}$ (e^+e^- -based) (Ref. [75]).

To evaluate the complete SM prediction of the muon anomalous magnetic moment a_μ one has also to account for the QED contribution $a_\mu^{\text{QED}} = (11658471.8951 \pm 0.0080) \times 10^{-10}$ [76], the electroweak contribution $a_\mu^{\text{EW}} = (15.36 \pm 0.10) \times 10^{-10}$ [77], as well as the higher-order $a_\mu^{\text{HHO}} = (-9.84 \pm 0.07) \times 10^{-10}$ [73] and light-by-light $a_\mu^{\text{Hbl}} = (11.6 \pm 4.0) \times 10^{-10}$ [78] hadronic contributions, that, together with a_μ^{HLO} (24), leads to

$$a_\mu = (11659185.1 \pm 10.3) \times 10^{-10}. \quad (25)$$

The difference between this value and the Brookhaven E821 experimental measurement⁷ [67]

$$a_\mu^{\text{exp}} = (11659208.9 \pm 6.3) \times 10^{-10} \quad (26)$$

is $(23.8 \pm 12.1) \times 10^{-10}$, that corresponds to the discrepancy of two standard deviations. As one can infer from Fig. 3, the estimation of the muon anomalous magnetic moment a_μ (25) fairly agrees with its recent evaluations [73–75].

B. Electromagnetic fine structure constant

The electromagnetic running coupling $\alpha_{\text{em}}(q^2)$ plays a central role in a variety of issues of precision particle physics. The vacuum polarization effects screen the electric charge and make the electromagnetic coupling α_{em} dependent on the energy scale q^2 :

$$\alpha_{\text{em}}(q^2) = \frac{\alpha}{1 - \Delta\alpha_{\text{lep}}(q^2) - \Delta\alpha_{\text{had}}(q^2)}, \quad (27)$$

with $\alpha = e^2/(4\pi) \simeq 1/137.036$ being the fine structure constant. In Eq. (27) the leptonic contribution $\Delta\alpha_{\text{lep}}(q^2)$ can reliably be calculated by making use of perturbation theory [80]. However, similarly to the aforementioned case of the muon anomalous magnetic moment, the hadronic contribution to Eq. (27)

$$\Delta\alpha_{\text{had}}(q^2) = -\frac{\alpha}{3\pi} q^2 \mathcal{P} \int_{m^2}^{\infty} \frac{R(s)}{s - q^2} \frac{ds}{s}, \quad (28)$$

(\mathcal{P} stands for the “Cauchy principal value”) involves the integration over the low-energy range and constitutes the prevalent source of the uncertainty of $\alpha_{\text{em}}(q^2)$, see discussion of this issue in, e.g., papers [73, 81] and references therein.

⁷ The averaged experimental value (26) accounts for the recently updated ratio of the muon-to-proton magnetic moment, see Ref. [79].

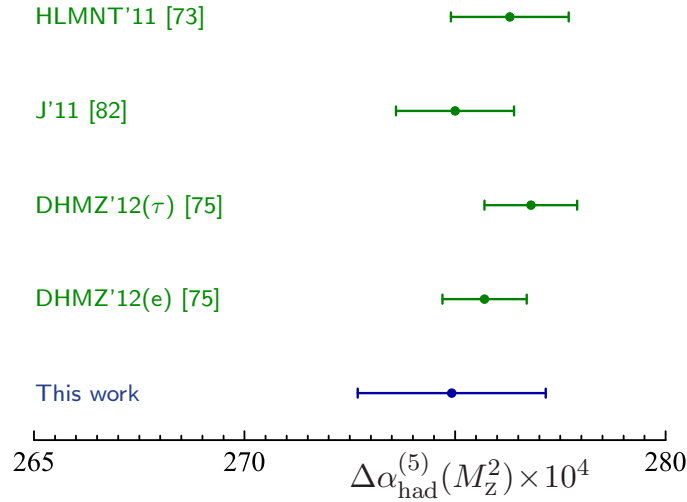


FIG. 4. Comparison of the hadronic contribution to the shift of the electromagnetic fine structure constant at the scale of Z boson mass (29) with its recent evaluations [73, 75, 82].

To evaluate the five-flavour⁸ hadronic contribution to the shift of the electromagnetic fine structure constant at the scale of Z boson mass in the framework of dispersive approach we shall follow the same lines as in Sec. IV A, that eventually yields

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = (274.9 \pm 2.2) \times 10^{-4}. \quad (29)$$

This equation corresponds to the four-loop level and the quoted error accounts for the uncertainties of the parameters entering Eq. (28), their values being taken from Refs. [71, 72]. The obtained estimation of $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ (29) appears to be in a good agreement with its recent evaluations, specifically, $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = (276.3 \pm 1.4) \times 10^{-4}$ (Ref. [73]), $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = (275.0 \pm 1.4) \times 10^{-4}$ (Ref. [82]), $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = (276.8 \pm 1.1) \times 10^{-4}$ (τ -based) and $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = (275.7 \pm 1.0) \times 10^{-4}$ (e^+e^- -based) (Ref. [75]), see Fig. 4. At the same time, Eq. (29) together with leptonic $\Delta\alpha_{\text{lep}}(M_Z^2) = (314.979 \pm 0.002) \times 10^{-4}$ [80] and top quark $\Delta\alpha_{\text{had}}^{\text{top}}(M_Z^2) = (-0.70 \pm 0.05) \times 10^{-4}$ [83] contributions lead to $\alpha_{\text{em}}^{-1}(M_Z^2) = 128.962 \pm 0.030$, that also agrees with recent assessments of this quantity, namely, $\alpha_{\text{em}}^{-1}(M_Z^2) = 128.962 \pm 0.018$ (Ref. [82]), $\alpha_{\text{em}}^{-1}(M_Z^2) = 128.944 \pm 0.019$ (Ref. [73]), and $\alpha_{\text{em}}^{-1}(M_Z^2) = 128.952 \pm 0.014$ (Ref. [75]).

It is worthwhile to mention also that the hadronic parts of the aforementioned electroweak observables (22) and (28) receive dominant contributions from different energy ranges. Specifically, the kernel in Eq. (28) signifies that sizable contributions come from the integration over low and intermediate energies, whereas the kernel in Eq. (22) indicates that a dominant contribution comes from the integration over the infrared domain. In particular, the latter implies that if the experimental data on $R(s)$ (or its phenomenological approximation) are involved into the evaluation of a_μ^{HLO} , then the contribution to Eq. (22) from the integration over the range of the lowest lying vector mesons is enhanced. As one might also note, the spectral function (15) contains only perturbative input. Nonetheless, $\rho_{\text{pert}}(\sigma)$ (15) appears to be efficient in the description of the quantities, which can be expressed as convolution of $R(s)$ and a respective kernel over a semi-infinite range. In particular, the spectral

⁸ The respective contribution of the top quark is, as usual, added separately, see Ref. [83].

function (15) is capable of describing the Adler function (10) (see Refs. [30, 31]), the hadronic vacuum polarization function (8) (see Sect. III), and yields the predictions for hadronic contributions to the aforementioned electroweak observables (22) and (28), which are in the right ballpark.

V. CONCLUSIONS

The hadronic vacuum polarization function obtained within dispersive approach contains no unphysical singularities and agrees with relevant lattice simulation data. The hadronic contributions to the muon anomalous magnetic moment and to the shift of the electromagnetic fine structure constant at the scale of Z boson mass estimated within dispersive approach conform with recent assessments of these quantities.

In further studies it would undoubtedly be interesting to include into the presented analysis the nonperturbative contributions arising from the operator product expansion and to explore possible constraints on the spectral density appearing in the approach on hand.

ACKNOWLEDGMENTS

The author is grateful to R. Kaminski, E. Passemar, M. Passera, J. Portoles, P. Roig, and H. Wittig for the stimulating discussions and useful comments.

Appendix: Correspondence between two sets of relations for $\Pi(q^2)$, $R(s)$, and $D(Q^2)$

As mentioned in Ref. [31], the integral representations (8)–(10) for the functions $\Pi(q^2)$, $R(s)$, and $D(Q^2)$ satisfy all six relations (2)–(7) by construction. It is straightforward to verify explicitly that the set of relations (2)–(7) holds for the leading-order terms (12)–(14) as well as for the most of the strong corrections (8)–(10). In particular, to show that the relations (3) and (6) are valid for the pair of the strong corrections [(8), (10)] one has to apply directly the integration and differentiation, respectively. To demonstrate that the relations (2) and (7) hold between pairs of expressions [(8), (9)] and [(9), (10)] the integration by parts is required. The validity of relation (5) for the pair [(9), (10)] can be shown by employing

$$\lim_{\varepsilon \rightarrow 0+} \frac{1}{x \pm i\varepsilon} = \mp i\pi\delta(x) + \mathcal{P}\frac{1}{x} \quad (\text{A.1})$$

in the respective integrand. The remaining relation (4) between the pair of the strong corrections [(8), (9)] is somewhat more laborious to demonstrate than the others, and will be addressed in this Section.

For the strong corrections $p(q^2)$ and $r(s)$ the relation (4) can be written as

$$r(s) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0+} \left[\Delta p(s + i\varepsilon, q_0^2) - \Delta p(s - i\varepsilon, q_0^2) \right], \quad (\text{A.2})$$

where $\Delta p(q^2, q_0^2) = p(q^2) - p(q_0^2)$. By virtue of Eq. (8)

$$\Delta p(s \pm i\varepsilon, q_0^2) = \int_{m^2}^{\infty} \rho(\sigma) \left[\ln \left(\frac{s - \sigma \pm i\varepsilon}{s - m^2 \pm i\varepsilon} \right) + \ln \left(\frac{m^2 - q_0^2}{\sigma - q_0^2} \right) \right] \frac{d\sigma}{\sigma}. \quad (\text{A.3})$$

Then, since

$$\lim_{\varepsilon \rightarrow 0_+} \ln(x \pm i\varepsilon) = \ln|x| \pm i\pi\theta(-x), \quad (\text{A.4})$$

the first term in the square brackets of Eq. (A.3) can eventually be represented as (the limit $\varepsilon \rightarrow 0_+$ is assumed hereinafter)

$$\ln \left(\frac{s - \sigma \pm i\varepsilon}{s - m^2 \pm i\varepsilon} \right) = \ln \left| \frac{s - \sigma}{s - m^2} \right| \pm i\pi\theta(s - m^2)\theta(\sigma - s). \quad (\text{A.5})$$

Thus, Eq. (A.3) acquires the form

$$\Delta p(s \pm i\varepsilon, q_0^2) = \int_{m^2}^{\infty} \rho(\sigma) \ln \left(\left| \frac{s - \sigma}{s - m^2} \right| \frac{m^2 - q_0^2}{\sigma - q_0^2} \right) \frac{d\sigma}{\sigma} \pm i\pi\theta(s - m^2) \int_s^{\infty} \rho(\sigma) \frac{d\sigma}{\sigma}, \quad (\text{A.6})$$

and, therefore, Eq. (A.2) reads

$$r(s) = \theta(s - m^2) \int_s^{\infty} \rho(\sigma) \frac{d\sigma}{\sigma}, \quad (\text{A.7})$$

that coincides with the integral representation (9).

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